ON SOME ECONOMIC MODELS SOLVABLE USING THE PARABOLIC MODEL OF THE LEAST SQUARES METHOD

CĂTĂLIN ILIE MITRAN *

ABSTRACT: *Many of today's techno-economic problems can be modeled by various mathematical methods. One of the best known and most applied is the so-called approximation tchnique by the method of the smallest squares.The best known caseofthis approximationmethod is the linear one.The present paper is dedicatedto another case of this method, namely the parabolic one.Initially, the main aspects related to the parabolic function and the method of the smallest squares will be presented, the parabolic case, after which a numerical example of calculation to exemplify the application of the method of the smallest squares, the parabolic case.Finally, two such applications of the mentioned method will be analyzed in approximation and estimation of the evolution of some economic indicators within the indicators dedicated to the circular economy indicators.*

KEY WORDS: *parabolic model, least squares method, economic indicators.*

JEL CLASSIFICATION: *J61.*

1. INTRODUCTION

We begin by recalling that a function of a the second degree , also called the parabolic function, is usually a function $f: \mathbb{R} \rightarrow \mathbb{R}$ which has the form

$$f(x) = ax^2 + bx + c \tag{1}$$

where $a, b, c \in \mathbb{R}$.

Various observations and measurements from various economic and technical fields can be described in the form of a lot of values of the form

* *Lecturer, Ph.D., University of Petrosani, Romania, catalinmitran@upet.ro*

$$\{(x_i, y_i), i = 1, \dots, n\} \tag{2}$$

where *n* represents the number of studies or research carried out, x_i generally represents the time intervals in which those studies and research were carried out and y_i the values obtained from them.

Definition 1 *The problem of determining a function f which, under the conditions given by (2), verifies the condition*

$$\sum_{i=1}^{n}(f(x_i) - y_i)^2 \to minim \tag{3}$$

is referred to as the smallest squares method.

Observation 1 *If in relation (3) the function* **(***3) is of form (1), then the latter is transformed into the relation*

$$\sum_{i=1}^{n}(f(x_i) - y_i)^2 = \sum_{i=1}^{n}(ax_i{}^2 + bx_i + c - y_i)^2 \to minim \tag{4}$$

We define

$$S(a, b. c) = \sum_{i=1}^{n}(ax_i{}^2 + bx_i + c - y_i)^2 \tag{5}$$

In this case we must find *a,b,c* so that

$$S(a, b. c) = \sum_{i=1}^{n}(ax_i{}^2 + bx_i + c - y_i)^2 \to minim \tag{6}$$

Wes shall briefly recall some essential results from mathematical analysis necessary for the issues addressed in this paper.

Let us consider $f: D \subseteq \mathbb{R}^n \to \mathbb{R}$, having the property that $f \in C^2(D)$, which means that function *f* admits all partial derivatives of the first and second order and these derivatives are all continuous.

Definition 2 *We say that point* $x_0 \in A$ *is a stationary point of function f if*

$$\begin{cases} \frac{\partial f}{\partial x_1}(x^0) = 0 \\ \frac{\partial f}{\partial x_2}(x^0) = 0 \\ \dots \\ \frac{\partial f}{\partial x_n}(x^0) = 0 \end{cases}. \tag{7}$$

where $\frac{\partial f}{\partial x_i}(x_0)$ means the partial derivative of the first order at the point x^0 in relation to the variable x_i .

Let us also consider the following values:

$$\Delta_1 = \frac{\partial^2 f}{\partial {x_1}^2}(x^0),\ \ \Delta_2 = \begin{vmatrix} \frac{\partial^2 f}{\partial {x_1}^2}(x^0) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x^0) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x^0) & \frac{\partial^2 f}{\partial {x_2}^2}(x^0) \end{vmatrix},$$

$$\Delta_3 = \begin{vmatrix} \frac{\partial^2 f}{\partial {x_1}^2}(x^0) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x^0) & \frac{\partial^2 f}{\partial x_1 \partial x_3}(x^0) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x^0) & \frac{\partial^2 f}{\partial {x_2}^2}(x^0) & \frac{\partial^2 f}{\partial x_2 \partial x_3}(x^0) \\ \frac{\partial^2 f}{\partial x_3 \partial x_1}(x^0) & \frac{\partial^2 f}{\partial x_3 \partial x_2}(x^0) & \frac{\partial^2 f}{\partial {x_3}^2}(x^0) \end{vmatrix}, \ldots,$$

$$\Delta_n = \begin{vmatrix} \frac{\partial^2 f}{\partial {x_1}^2}(x^0) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x^0) & \frac{\partial^2 f}{\partial x_1 \partial x_3}(x^0) & \ldots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(x^0) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x^0) & \frac{\partial^2 f}{\partial {x_2}^2}(x^0) & \frac{\partial^2 f}{\partial x_2 \partial x_3}(x^0) & \ldots & \frac{\partial^2 f}{\partial x_2 \partial x_n}(x^0) \\ \frac{\partial^2 f}{\partial x_3 \partial x_1}(x^0) & \frac{\partial^2 f}{\partial x_3 \partial x_2}(x^0) & \frac{\partial^2 f}{\partial {x_3}^2}(x^0) & \ldots & \frac{\partial^2 f}{\partial x_3 \partial x_n}(x^0) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(x^0) & \frac{\partial^2 f}{\partial x_n \partial x_2}(x^0) & \frac{\partial^2 f}{\partial x_n \partial x_3}(x^0) & \ldots & \frac{\partial^2 f}{\partial {x_n}^2}(x^0) \end{vmatrix}$$

where $\frac{\partial^2 f}{\partial x_i \partial x_j}(x^0)$ represents the partial derivative of the second order with respect to the variables x_i and x_j calculated at the point x^0 .

The following result related to the points of extreme, maximum or minimum,is given by the following theorem:

Theorem 1

For $f: D \subseteq \mathbb{R}^n \to \mathbb{R}$, having the property that $f \in C^2(D)$ and x^0 a stationary point the following results occur:

1) If $\Delta_1 > 0, \Delta_2 > 0, \Delta_3 > 0, \ldots, \Delta_n > 0$ then x^0 is a minimum point;
2) If $\Delta_1 < 0, \Delta_2 > 0, \Delta_3 < 0, \ldots, (-1)^n \Delta_n > 0$ then x^0 is a maximum point.

In our particular case we have as definition domain $\mathbb{R}^3$ instead of $\mathbb{R}^n$ and S instead of f and a, b, c instead of unknows x_1, x_2, x_3.Taking care of (7) we have to solve the system:

$$\begin{cases} \frac{\partial S}{\partial a} = 0 \Rightarrow 2\sum_{i=1}^{n}(a{x_i}^2 + bx_i + c - y_i){x_i}^2 = 0 \\ \frac{\partial S}{\partial b} = 0 \Rightarrow 2\sum_{i=1}^{n}(a{x_i}^2 + bx_i + c - y_i)x_i = 0 \\ \frac{\partial S}{\partial c} = 0 \Rightarrow 2\sum_{i=1}^{n}(a{x_i}^2 + bx_i + c - y_i) \cdot 1 = 0 \end{cases} \tag{8}$$

that is equivalent, dividing by two and doing the math with

$$\begin{cases} a\sum_{i=1}^{n} {x_i}^4 + b\sum_{i=1}^{n} {x_i}^3 + c\sum_{i=1}^{n} {x_i}^2 = \sum_{i=1}^{n} {x_i}^2 f(x_i) \\ a\sum_{i=1}^{n} {x_i}^3 + b\sum_{i=1}^{n} {x_i}^2 + c\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i f(x_i) \\ a\sum_{i=1}^{n} {x_i}^2 + b\sum_{i=1}^{n} x_i + c\cdot n = \sum_{i=1}^{n} f(x_i) \end{cases} \tag{9}$$

In order to solve this last system, we build the next table:

Table 1

i	x_i	${x_i}^2$	${x_i}^3$	${x_i}^4$	$f(x_i)$	$x_i f(x_i)$	${x_i}^2 f(x_i)$
1	x_1	${x_1}^2$	${x_1}^3$	${x_1}^4$	$f(x_1)$	$x_1 f(x_1)$	${x_1}^2 f(x_1)$
2	x_2	${x_2}^2$	${x_2}^3$	${x_2}^4$	$f(x_2)$	$x_2 f(x_2)$	${x_2}^2 f(x_2)$
…	…	…					
n	x_n	${x_n}^2$	${x_n}^3$	${x_n}^4$	$f(x_n)$	$x_n f(x_n)$	${x_n}^2 f(x_n)$
$\sum$	$\sum_{i=1}^{n} x_i$	$\sum_{i=1}^{n} {x_i}^2$	$\sum_{i=1}^{n} {x_i}^3$	$\sum_{i=1}^{n} {x_i}^4$	$\sum_{i=1}^{n} f(x_i)$	$\sum_{i=1}^{n} x_i f(x_i)$	$\sum_{i=1}^{n} {x_i}^2 f(x_i)$

It is possible to demonstrate that:

$$\Delta = \begin{vmatrix} \sum_{i-1}^{n} {x_i}^4 & \sum_{i=1}^{n} {x_i}^3 & \sum_{i=1}^{n} {x_i}^2 \\ \sum_{i=1}^{n} {x_i}^3 & \sum_{i=1}^{n} {x_i}^2 & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} {x_i}^2 & \sum_{i=1}^{n} x_i & n \end{vmatrix} \neq 0$$

In this case we have a one-stop solution which can be calculated using the so-called Cramer method:

$$a = \frac{\Delta_a}{\Delta}, b = \frac{\Delta_b}{\Delta}, c = \frac{\Delta_c}{\Delta}, \tag{10}$$

where

$$\Delta_a = \begin{vmatrix} \sum_{i=1}^{n} {x_i}^2 f(x_i) & \sum_{i=1}^{n} {x_i}^3 & \sum_{i=1}^{n} {x_i}^2 \\ \sum_{i=1}^{n} x_i f(x_i) & \sum_{i=1}^{n} {x_i}^2 & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} f(x_i) & \sum_{i=1}^{n} x_i & n \end{vmatrix}$$
$$\Delta_b = \begin{vmatrix} \sum_{i=1}^{n} {x_i}^4 & \sum_{i=1}^{n} {x_i}^2 f(x_i) & \sum_{i=1}^{n} {x_i}^2 \\ \sum_{i=1}^{n} {x_i}^3 & \sum_{i=1}^{n} x_i f(x_i) & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} {x_i}^2 & \sum_{i=1}^{n} f(x_i) & n \end{vmatrix} \tag{11}$$
$$\Delta_c = \begin{vmatrix} \sum_{i-1}^{n} {x_i}^4 & \sum_{i=1}^{n} {x_i}^4 & \sum_{i=1}^{n} {x_i}^2 f(x_i) \\ \sum_{i=1}^{n} {x_i}^3 & \sum_{i=1}^{n} {x_i}^2 & \sum_{i=1}^{n} x_i f(x_i) \\ \sum_{i=1}^{n} {x_i}^2 & \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} f(x_i) \end{vmatrix}$$

Let us consider a^*, b^*, c^* as being the solution of the system (9). In order to demonstrate that (a^*, b^*, c^*) is a minimum point for $S(a, b.c)$ we have to demonstrate that $\Delta_1 = \frac{\partial^2 S}{\partial a^2}(a^*, b^*, c^*) > 0$,

$$\Delta_2 = \begin{vmatrix} \frac{\partial^2 S}{\partial a^2}(a^*, b^*, c^*) & \frac{\partial^2 S}{\partial a \partial b}(a^*, b^*, c^*) \\ \frac{\partial^2 S}{\partial b \partial a}(a^*, b^*, c^*) & \frac{\partial^2 S}{\partial b^2}(a^*, b^*, c^*) \end{vmatrix} > 0$$

and

$$\Delta_3 = \begin{vmatrix} \frac{\partial^2 S}{\partial a^2}(a^*, b^*, c^*) & \frac{\partial^2 S}{\partial a \partial b}(a^*, b^*, c^*) & \frac{\partial^2 S}{\partial a \partial c}(a^*, b^*, c^*) \\ \frac{\partial^2 S}{\partial b \partial a}(a^*, b^*, c^*) & \frac{\partial^2 S}{\partial b^2}(a^*, b^*, c^*) & \frac{\partial^2 S}{\partial b \partial c}(a^*, b^*, c^*) \\ \frac{\partial^2 S}{\partial c \partial a}(a^*, b^*, c^*) & \frac{\partial^2 S}{\partial c \partial b}(a^*, b^*, c^*) & \frac{\partial^2 S}{\partial c^2}(a^*, b^*, c^*) \end{vmatrix} > 0 .$$

It can be seen immediately that

$$\Delta_1 = \frac{\partial^2 S}{\partial a^2}(a^*, b^*, c^*) = 4\sum_{i=1}^{n} {x_i}^2 > 0$$

For Δ_2 and Δ_3 we have the next values:

$$\Delta_2 = 4 \cdot \begin{vmatrix} \sum_{i=1}^{n} {x_i}^4 & \sum_{i=1}^{n} {x_i}^3 \\ \sum_{i=1}^{n} {x_i}^3 & \sum_{i=1}^{n} {x_i}^2 \end{vmatrix} \text{ and } \Delta_3 = 8 \cdot \begin{vmatrix} \sum_{i=1}^{n} {x_i}^4 & \sum_{i=1}^{n} {x_i}^3 & \sum_{i=1}^{n} {x_i}^2 \\ \sum_{i=1}^{n} {x_i}^3 & \sum_{i=1}^{n} {x_i}^2 & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} {x_i}^2 & \sum_{i=1}^{n} x_i & n \end{vmatrix}$$

With regard to Δ_2 and Δ_3 it is not so simple to show that they have strictly positive values, the demonstration of this fact requiring complex calculation, which is why it was abandoned, admitting, in the end, only the result.

2. NUMERICAL EXAMPLE

The evolution of a certain company for a certain category of products has been studied monthly for a year, obtaining the following monthly results related to receipts:

1) In June: 5735 currency units;
2) In July: 5375 currency units;
3) In August: 4624 currency units;
4) In September: 3827 currency units;
5) In October: 3263 currency units;
6) In November: 2935 currency units;
7) In December: 2743 currency units;
8) In January: 3325 currency units;
9) In February: 3976 currency units;
10) In March: 4433 currency units;
11) In Aprilie: 5117 currency units;
12) In May: 5425 currency units.

As you can see, the sales figure is higher in the summer months and then decreases towards autumn and winter, after which it begins to increase again in spring. We shall organize the data so that June will have the index 1, July index 2 and so on, thus obtaining table 2:

Table 2

Number	Thousands of monetary units
1	5.735
2	5.375
3	4.624
4	3.827
5	3.263
6	2.935
7	2.743
8	3.325
9	3.976
10	4.433
11	5.117
12	5.425

Taking into account the data presented it can be seen relatively easily that they would fit into a parabolic distribution-type model.

Table 1 will look like this in our case:

Table 3

i	x_i	${x_i}^2$	${x_i}^3$	${x_i}^4$	$f(x_i)$	$x_i f(x_i)$	${x_i}^2 f(x_i)$
1	1	1	1	1	$f(x_1)$	$x_1 f(x_1)$	${x_1}^2 f(x_1)$
2	2	4	8	16	$f(x_2)$	$x_2 f(x_2)$	${x_2}^2 f(x_2)$
3	3	9	27	81	$f(x_3)$	$x_3 f(x_3)$	${x_3}^2 f(x_3)$
4	4	16	64	256	$f(x_4)$	$x_4 f(x_4)$	${x_4}^2 f(x_4)$
5	5	25	125	625	$f(x_5)$	$x_5 f(x_5)$	${x_5}^2 f(x_5)$
6	6	36	216	1296	$f(x_6)$	$x_6 f(x_6)$	${x_6}^2 f(x_6)$
7	7	49	343	2401	$f(x_7)$	$x_7 f(x_7)$	${x_7}^2 f(x_7)$
8	8	64	512	4096	$f(x_8)$	$x_8 f(x_8)$	${x_8}^2 f(x_8)$
9	9	81	729	6561	$f(x_9)$	$x_9 f(x_9)$	${x_9}^2 f(x_9)$
10	10	100	1000	10000	$f(x_{10})$	$x_{10} f(x_{10})$	${x_{10}}^2 f(x_{10})$
11	11	121	1331	14641	$f(x_{11})$	$x_{11} f(x_{11})$	${x_{11}}^2 f(x_{11})$
12	12	144	1728	20736	$f(x_{12})$	$x_{12} f(x_{12})$	${x_{12}}^2 f(x_{12})$
Σ	78	650	6084	60710	$\sum_{i=1}^{12} f(x_i)$	$\sum_{i=1}^{12} x_i f(x_i)$	$\sum_{i=1}^{12} {x_i}^2 f(x_i)$

and by making the appropriate replacements the following data are obtained, as can be seen in table 4:

Table 4

i	x_i	$x_i{}^2$	$x_i{}^3$	$x_i{}^4$	$f(x_i)$	$x_i f(x_i)$	$x_i{}^2 f(x_i)$
1	1	1	1	1	5,735	5,375	5,375
2	2	4	8	16	5,375	10,75	21,5
3	3	9	27	81	4,624	13,872	41,616
4	4	16	64	256	3,827	15,308	61,232
5	5	25	125	625	3,263	16,315	81,575
6	6	36	216	1296	2,935	17,61	105,66
7	7	49	343	2401	2,743	19,201	134,407
8	8	64	512	4096	3,325	26,6	212,8
9	9	81	729	6561	3,976	35,784	322,056
10	10	100	1000	10000	4,433	44,33	443,3
11	11	121	1331	14641	5,117	56,287	619,157
12	12	144	1728	20736	5,425	65,1	781,2
Σ	78	650	6084	60710	50,778	326,892	2830,238

The system begins:

$$\begin{cases} 60710 \cdot a + 6084 \cdot b + 650 \cdot c = 2830{,}238 \\ 6084 \cdot a + 650 \cdot b + 78 \cdot c = 326{,}892 \\ 650 \cdot a + 78 \cdot b + 12 \cdot c = 50{,}778 \end{cases}$$

The determinant of the system is:

$$\begin{vmatrix} 60710 & 6084 & 650 \\ 6084 & 650 & 78 \\ 650 & 78 & 12 \end{vmatrix} = 2290288 \neq 0.$$

In this case, taking into account relations (10) and (11) we shall have

$$\Delta_a = \begin{vmatrix} 2830{,}238 & 6084 & 650 \\ 326{,}892 & 650 & 78 \\ 50{,}778 & 78 & 12 \end{vmatrix} = 207428{,}128$$

and

$$a = \frac{\Delta_a}{\Delta} = \frac{207.428{,}128}{2.290.288} \approx 0{,}0906\ ,$$

$$\Delta_b = \begin{vmatrix} 60710 & 2830{,}238 & 650 \\ 6084 & 326{,}892 & 78 \\ 650 & 50{,}778 & 12 \end{vmatrix} = -2.924.796{,}568$$

and

$$b = \frac{\Delta_b}{\Delta} = \frac{-2.924.796{,}568}{2.290.288} \approx -1{,}2767\ ,$$

$$\Delta_c = \begin{vmatrix} 60710 & 6084 & 2830{,}238 \\ 6084 & 650 & 326{,}892 \\ 650 & 78 & 50{,}778 \end{vmatrix} = 16.314.346{,}048$$

and

$$c = \frac{\Delta_c}{\Delta} = \frac{16.314.346{,}048}{2.290.288} \approx 7{,}123 \ .$$

With these values thus obtained, the approximation will be of the form:

$$y(x) = 0{,}0906 \cdot x^2 - 1{,}2767 \cdot x + 7{,}123 \ . \qquad (12)$$

Remark 1 The example presented is intended to be only one that shows how the approximate function is actually determined, which is considered to be a parabolic type. A rigorous analysis of this fact can be made using rigorous statistical-mathematical criteria of analysis regarding the veracity of the model.

Notable results related to the approximation by the method of the smallest squares, the parabolic cases, were obtained in (Dobre-Baron, et al., 2022). These results are related to the study of certain economic indicators belonging to the *Circular Economy* field. These indicators can be consulted in the so-called EUROSTAT tables (https://ec.europa.eu/eurostat). This study is applied to certain economic in certain broader areas of interests.

In the case of the area called *Production and consumption* it was analyzed the indicator called *Generation of waste excluding major mineral wastes per domestic material consumption.* For this case the time period during 2004 and 2018 was taken into account, with the account that the statistical data are at two-years intervals. The data are presented in the next table:

Table 5

Year	Romania(%)
2004	13,2
2006	13,6
2008	10,3
2010	8,6
2012	6,5
2014	5,6
2016	4,7
2018	4,8

where the parabolic model's parameters are:
$a = 0{,}0313, b = -126{,}99, c = 128421$.

In this case the the parabolic regression will be

$$y(x) = 0{,}0313 \cdot x^2 - 126{,}99 \cdot x + 128421 \ . \qquad (13)$$

This approximation also allows us to make the following forecast:

Table 6

Year	2020	2022	2024
Romania (%)	3,89	3,72	3,71

Also in the case of the area called *Competitiveness and innovation* it was analyzed the indicator called *Gross investment in tangible goods – percentage of gross domestic product(GDP).* A parabolic estimate can be made for the data presented in table 7:

Table 7

Year	UE-27(%)
2004	0,13
2006	0,12
2008	0,1
2010	0,1
2012	0,11
2014	0,11
2016	0,12
2018	0,12

where the parabolic model's parameters are $a = 0{,}00046, b = -1{,}85, c = 1866{,}087$.

We can thus make the estimate:

$$y(x) = 0{,}00046 \cdot x^2 - 1{,}85 \cdot x + 1866{,}087 \tag{14}$$

This estimate is also followed by the forecast

Table 8

Year	2020	2022	2024
UE (%)	3,89	3,72	3,71

3. CONCLUSIONS

Application of the least squares method, the parabolic case in the case of the those indicators, the “Generation of waste excluding major mineral wastes per domestic material consumption” considered in the case of Romania and the “Gross investment in tangible goods – percentage of gross domestic product (GDP)” considered in the case of UE-27 allows the subsequent estimate of the evolution of thr phenomenadescribed by the respective indicators, at least for a subsequent limited period of time, in the cases presented being three units of time from those used in those indicators.

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